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Exploring the new soliton solutions to the nonlinear M-fractional evolution equations in shallow water by three analytical techniques

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ABSTRACT

This paper explores the new soliton solutions of the evolution equations named as truncated M-fractional (1+1)-dimensional non-linear Kaup-Boussinesq system by utilizing the $\exp_a$ function, modified simplest equation and Sardar sub-equation techniques. This system is used in the analysis of long waves in shallow water. The attained results involving trigonometric, hyperbolic and exponential functions. The effect of fractional order derivative is also discussed. Obtained results are very close to the approximate results due to the use of M-fractional derivative. Achieved results are verified by Mathematica tool. Few of the gained results are also explained through 2-D, 3-D and contour graphs. At the end, these techniques are straight forward, useful and effective to deal with non-linear FPDEs.

Introduction

Soliton theory based on water waves, plasmas, optical fibers etc., was developed in 1960–1970. This is significant branch of applied mathematics as well as mathematical physics. It has significant uses in non-linear optics, fluid mechanics, plasmas etc. This theory is widely applied in various natural sciences, including communication, biology, chemistry, mathematics and almost all branches of physics including fluid dynamics, condensed matter physics, plasma physics etc. Distinct kinds of naturally occurring phenomenon are expressed in the form of non-linear evolution models. Many schemes are made to achieve exact wave results for evolution equations. For example, Fractional sub-equation technique [1], Variable separation scheme [2], Inverse scattering technique [3], Riemann–Hilbert method [4], Darboux transformation technique [5], Extended simple equation technique [6], ϕ^6-model expansion scheme [7], Simplified extended tanh-function technique [8], Conventional Khater method [9], Improved $\tan(\frac{\psi(\eta)}{2})$ [10], Paul–Painlevé method [11,12], Laplace homotopy perturbation scheme [13] and many more.

Our study model is the important evolution equations named as truncated M-fractional Kaup-Boussinesq system. Various kinds of analytical solitons are achieved through distinct techniques in literature. For example, some exact traveling solutions are gained by utilizing auxiliary equation scheme [14], the trigonometric function, rational function, exponential function and jacobian function solutions are achieved by using complete discriminant system technique of polynomial [15], new analytical solitons are gained by utilizing first integral scheme [16].

In our study we use three simple, useful and significant techniques named as $\exp_a$ function technique, modified simplest equation technique and Sardar sub-equation technique. The general $\exp_a$ function scheme was first time proposed by Ahmed T.Ali and Ezzat R. Hassan in 2010 [17]. There are various uses of these techniques. Instantly; optical wave solutions of perturbed Gerdjikov–Ivanov model by utilizing the $\exp_a$ function scheme [18], some new optical solitons of Sasa–Satsuma higher order equation in [19], the dark soliton, bright soliton and combo optical solitons of three coupled Maccari's system are achieved [20], the dark, bright and many other exact results of modified Camassa–Holm model are obtained in [21]. Some solitary wave solutions of BBM and Chan–Hilliard equations by utilizing modified simplest equation method [22], exact wave solutions of Boussinesq

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and coupled Boussinesq equations have been achieved by using this technique in [23]. Similarly, some exact wave solutions of new Hamiltonian Amplitude equation are attained by using Sardar sub-equation technique [24].

Basic aim of our research is to explore new soliton solutions of (1+1)-dimensional non-linear Kaup-Boussinesq system with the help of $\exp_a$ function technique, modified simplest equation technique and Sardar sub-equation technique and the effect of truncated M-fractional derivative.

In 1975, Kaup considered the water waves propagating in an infinite narrow channel of constant mean depth by adding one more order of nonlinearity in the derivation of Boussinesq equation from the shallow water wave model and proposed the Kaup–Boussinesq (KB) equation, whose inverse scattering problem has energy dependent Schrödinger potential. The KB equation has not only good mathematical properties but also rich physical meanings, which has attracted many researchers. In our research, we use truncated M-fractional derivative that fulfill the both characteristics of integer and fractional order derivatives. The $\exp_a$ function technique is used to obtain the optical wave solutions. The modified simplest equation technique capable to lead to solutions of non-linear PDEs that are more complicated than a single solitary wave. This technique is important for finding the exact traveling wave solutions of non-linear evolution equations in mathematical physics. The benefit of Sardar sub-equation technique is that it provides a wide variety of soliton solutions having dark, bright, singular, periodic singular as well as combined dark-singular and combined dark-bright solitons.

The paper contains different sections; In Section “Model description and mathematical analysis”: we describe the governing system and its mathematical treatment. In Section “The $\exp_a$ function technique”: we describe $\exp_a$ function technique and its application to obtain the soliton solutions. In Section “Modified simplest equation technique”: we explain the modified simplest equation technique and apply it to achieve the new soliton solutions. In Section “Description of Sardar sub-equation Method”: we describe Sardar sub-equation technique and its application to obtain the soliton solutions. In Section “New exact wave solutions through Sardar sub-equation Technique”: we describe some gained results by 2-D, 3-D and Contour plots. In Section “Graphically interpretation of results”: we mention conclusion.

Model description and mathematical analysis

Assume the space–time fractional (1+1)-dimensional Kaup–Boussinesq system given as [14].

$$D^{\alpha,Y}_{M,t}h - D^{3\alpha,Y}_{M,3x}g - 2D^{\alpha,Y}_{M,x}(hg) = 0.$$
$$D^{\alpha,Y}_{M,t}g - D^{\alpha,Y}_{M,x}h - D^{\alpha,Y}_{M,x}(g^2) = 0. \quad (1)$$

where

$$D^{\alpha,Y}_{M,x}h(x) = \lim_{\tau\to 0}\frac{h\,(x\ E_Y(\tau x^{1-\alpha})) - h(x)}{\tau}, \quad \alpha \in (0,1],\ \ Y > 0, \quad (2)$$

Here $E_Y(.)$ denotes truncated Mittag-Leffler (TML) function shown in [25,26]. This definition of derivative is a generalization of all derivatives and the other definitions are the special cases of this. This definition is valid for all the models and also for our under study model.

here h=h(x,t) represents the water surface height above a horizontal bottom and g=g(x, t) denotes the horizontal velocity field.

Assume the following wave transformation:

$$h(x,t) = H(\xi),\ g(x,t) = G(\xi),\ \xi = \frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \delta t^\alpha)\ . \quad (3)$$

By inserting Eq. (3) into the Eq. (1), we get the NLODE system shown as:

$$\delta H' + G''' + 2GH' + 2HG' = 0.$$
$$\delta G' + H' + 2GH' = 0. \quad (4)$$

Integrating the second equation of the above system once w.t.r ξ and taking integration constant zero, we get,

$$H = -\delta G - G^2. \quad (5)$$

Putting Eq. (5) into the first equation of the system (4) and integrating it once w.t.r ξ, we gain

$$G'' - \delta^2 G - 3\delta G^2 - 2G^3 = 0. \quad (6)$$

By using homogeneous balance scheme, we get m=1. We will explore the new soliton solutions of Eq. (6) by applying two different techniques in the following.

The $\exp_a$ function technique

We describe the basic steps of this technique.

Let us take the NLPDE;

$$F(f, f^2 f_y, f_t, f_{xy}, f_{tt}, f_{yt}, \ldots) = 0. \quad (7)$$

Eq. (7) changed into NLODE:

$$Y(F, F', F'', \ldots,) = 0. \quad (8)$$

By applying the below wave transformation:

$$f(x,\ y,\ t) = F(\xi), \xi = \delta x + \mu y + \lambda t. \quad (9)$$

Considering the solutions of Eq. (8) are shown as [27–30]:

$$F(\xi) = \frac{\alpha_0 + \alpha_1 d^\xi + \ldots + \alpha_m d^{m\xi}}{\beta_0 + \beta_1 d^\xi + \cdots + \beta_m d^{m\xi}},\ d \neq 0, 1. \quad (10)$$

here α_s and $\beta_s (0 \le s \le m)$ are undetermined. Positive integer m is found by homogeneous balance method on Eq. (10). Inserting Eq. (10) into the Eq. (8), yields

$$\wp(d^\xi) = \ell_0 + \ell_1 d^\xi + \cdots + \ell_t d^{t\xi} = 0. \quad (11)$$

Inserting ℓ_s $(0 \le s \le t)$ in Eq. (11) equal to 0, a set of algebraic equations is attained given as

$$\ell_s = 0,\ here\ s = 0, \ldots, t. \quad (12)$$

By using the obtain solutions, one can attain the soliton solutions for Eq. (7).

New wave solutions by the $\exp_a$ function technique

Eq. (10) reduces into below form for m=1:

$$U(\xi) = \frac{\alpha_0 + \alpha_1 d^\xi}{\beta_0 + \beta_1 d^\xi} \quad (13)$$

Substituting Eq. (13) into Eq. (6), a system is gained. Manipulating the obtained system, we get distinct results:

Set 1:

$$\{\alpha_0 = 0, \alpha_1 = \beta_1 \log(d), \delta = -\log(d)\} \quad (14)$$

$$g(x,t) = \frac{\beta_1 \log(d)\ d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}}{\beta_0 + \beta_1\ d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}} \quad (15)$$

$$h(x,t) = \log(d)\left(\frac{\beta_1 \log(d)\ d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}}\right) - \left(\frac{\beta_1 \log(d)\ d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}}\right)^2 \quad (16)$$

Set 2:

$$\{\alpha_0 = \beta_0 \log(d), \alpha_1 = 0, \delta = -\log(d)\} \quad (17)$$

$$g(x,t) = \frac{\beta_0 \log(d)}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}} \quad (18)$$

$$h(x,t) = \log(d)(\frac{\beta_0 \log(d)}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}}) - (\frac{\beta_0 \log(d)}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha + \log(d)\ t^\alpha))}})^2 \quad (19)$$

Set 3:

$$\{\alpha_0 = 0, \alpha_1 = -\beta_1 \log(d), \delta = \log(d)\} \quad (20)$$

$$g(\xi,t) = -\frac{\beta_1 \log(d) d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}} \quad (21)$$

$$h(x,t) = -\log(d)(-\frac{\beta_1 \log(d) d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}}) - (-\frac{\beta_1 \log(d) d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}})^2 \quad (22)$$

Set 4:

$$\{\alpha_0 = -\beta_0 \log(d), \alpha_1 = 0, \delta = \log(d)\} \quad (23)$$

$$g(x,t) = -\frac{\beta_0 \log(d)}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}} \quad (24)$$

$$h(x,t) = -\log(d)(-\frac{\beta_0 \log(d)}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}}) - (-\frac{\beta_0 \log(d)}{\beta_0 + \beta_1 d^{(\frac{\Gamma(1+Y)}{\alpha}(x^\alpha - \log(d)t^\alpha))}})^2 \quad (25)$$

Modified simplest equation technique

Here we describe the some basic steps of the technique:

Step 1:

Assuming the NLPDE:

$$G(f, f^2, f^2 f_x, f_y, f_{yy}, f_{xx}, f_{xy}, f_{xt}, \dots) = 0, \quad (26)$$

here $f = f(x,y,t)$ represents the wave profile.

Consider the traveling wave transformation:

$$f(x,y,t) = F(\xi), \ \ \xi = x - \mu y + \lambda t. \quad (27)$$

Inserting the Eq. (27) into the Eq. (26), we attain NLODE:

$$V(F(\xi), F^2(\xi)F'(\xi), F''(\xi), \dots) = 0. \quad (28)$$

Step 2:Consider Eq. (28) has the following solution form:

$$F(\xi) = \sum_{j=1}^{m} b_j \psi^j(\xi) \quad (29)$$

here $b_j (j = 1, 2, \dots, m)$ are undetermined and $b_m \neq 0$. A new profile $\psi(\xi)$ fulfill below ODE:

$$\psi'(\xi) = \psi^2(\xi) + \omega \quad (30)$$

where ω is a parameter.

Eq. (30) have solutions for different cases of ω:

If $\omega < 0$,

$$\psi(\xi) = -\sqrt{-\omega}\tanh(\sqrt{-\omega}\xi) \quad (31)$$

$$\psi(\xi) = -\sqrt{-\omega}\coth(\sqrt{-\omega}\xi) \quad (32)$$

$$\psi(\xi) = \sqrt{-\omega}(-\tanh(2\sqrt{-\omega}\ \xi) \pm i sech(2\sqrt{-\omega}\ \xi)), \quad (33)$$

$$\psi(\xi) = \sqrt{-\omega}(-\coth(2\sqrt{-\omega}\ \xi) \pm csch(2\sqrt{-\omega}\ \xi)), \quad (34)$$

$$\psi(\xi) = -\frac{\sqrt{-\omega}}{2}(\tanh(\frac{\sqrt{-\omega}}{2}\ \xi) + \coth(\frac{\sqrt{-\omega}}{2}\ \xi)). \quad (35)$$

If $\omega > 0$,

$$\psi(\xi) = \sqrt{\omega}\tan(\sqrt{\omega}\xi) \quad (36)$$

$$\psi(\xi) = -\sqrt{\omega}\cot(\sqrt{\omega}\xi) \quad (37)$$

$$\psi(\xi) = \sqrt{\omega}(\tan(2\sqrt{\omega}\ \xi) \pm \sec(2\sqrt{\omega}\ \xi)), \quad (38)$$

$$\psi(\xi) = \sqrt{\omega}(-\cot(2\sqrt{\omega}\ \xi) \pm \csc(2\sqrt{\omega}\ \xi)), \quad (39)$$

$$\psi(\xi) = \frac{\sqrt{\omega}}{2}(\tan(\frac{\sqrt{\omega}}{2}\ \xi) - \cot(\frac{\sqrt{\omega}}{2}\ \xi)). \quad (40)$$

If $\omega = 0$,

$$\psi(\xi) = -\frac{1}{\xi} \quad (41)$$

Step 3: Using Eq. (29) into the Eq. (28) with Eq. (30) and collecting the co-efficients of every power of ψ^j. Substituting each of them equal to zero, we obtain a set of equations involving b_i, λ, μ. Solving the gained set of equations, we gain results for parameters.
Step 4: Inserting Eq. (28) of which b_j, λ, μ has been obtained into Eq. (29), we gain analytical wave solutions of Eq. (26).

New soliton solutions of Eq. (6) by MSET

Eq. (29) reduces into shown form for m = 1:

$$G(\xi) = b_0 + b_1\psi(\xi) \quad (42)$$

Using Eq. (42) into Eq. (6) with Eq. (30). Solving the achieved system by Mathematica tool, we obtain solution.

Set 1:

$$\{b_0 = b_0, b_1 = -1, \delta = -2b_0, \omega = -b_0^2\} \quad (43)$$

Case 1: If $\omega < 0$,

$$g(x,\ t) = b_0 + \sqrt{-\omega}\tanh(\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha)) \quad (44)$$

$$h(x,t) = 2b_0(b_0 + \sqrt{-\omega}\tanh(\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha))) - (b_0 + \sqrt{-\omega}\tanh(\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha)))^2 \quad (45)$$

$$g(x,t) = b_0 + \sqrt{-\omega}\coth(\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha)) \quad (46)$$

$$h(x,t) = 2b_0(b_0 + \sqrt{-\omega}\coth(\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha))) - (b_0 + \sqrt{-\omega}\coth(\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha)))^2 \quad (47)$$

$$g(x,t) = b_0 - \sqrt{-\omega}(\{-\tanh(2\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha)) \pm i sech(2\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha))\}), \quad (48)$$

$$h(x,t) = 2b_0(b_0 - \sqrt{-\omega}(\{-\tanh(2\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha)) \pm i sech(2\sqrt{-\omega}\ \frac{\Gamma(1+Y)}{\alpha}(x^\alpha + 2b_0 t^\alpha))\}))$$

$$-(b_0-\sqrt{-\omega}(\{-\tanh(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm isech(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}))^2, \quad (49)$$

$$g(x,t)=b_0-\sqrt{-\omega}(\{-\coth(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm \operatorname{csch}(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}), \quad (50)$$

$$h(x,t) = 2b_0(b_0-\sqrt{-\omega}(\{-\coth(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm \operatorname{csch}(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\})) -(b_0-\sqrt{-\omega}(\{-\coth(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm \operatorname{csch}(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}))^2, \quad (51)$$

$$g(x,t)=b_0+\frac{\sqrt{-\omega}}{2}(\{\tanh(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) + \coth(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}) \quad (52)$$

$$h(x,t) = 2b_0(b_0+\frac{\sqrt{-\omega}}{2}(\{\tanh(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) + \coth(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\})) -(b_0+\frac{\sqrt{-\omega}}{2}(\{\tanh(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) + \coth(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}))^2 \quad (53)$$

Set 2:

$$\left\{b_0=b_0, b_1=1, \delta=-2b_0, \omega=-b_0^2\right\} \quad (54)$$

Case 1: If $\omega<0$,

$$g(x,t)=b_0-\sqrt{-\omega}\tanh(\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \quad (55)$$

$$h(x,t)=2b_0(b_0-\sqrt{-\omega}\tanh(\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))) - (b_0-\sqrt{-\omega}\tanh(\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)))^2 \quad (56)$$

$$g(x,t)=b_0-\sqrt{-\omega}\coth(\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \quad (57)$$

$$h(x,t)=2b_0(b_0-\sqrt{-\omega}\coth(\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))) - (b_0-\sqrt{-\omega}\coth(\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)))^2 \quad (58)$$

$$g(x,t)=b_0+\sqrt{-\omega}(\{-\tanh(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm isech(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}), \quad (59)$$

$$h(x,t) = 2b_0(b_0+\sqrt{-\omega}(\{-\tanh(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm isech(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\})) -(b_0+\sqrt{-\omega}(\{-\tanh(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm isech(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}))^2, \quad (60)$$

$$g(x,t)=b_0+\sqrt{-\omega}(\{-\coth(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm \operatorname{csch}(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}), \quad (61)$$

$$h(x,t) = 2b_0(b_0+\sqrt{-\omega}(\{-\coth(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm \operatorname{csch}(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\})) -(b_0+\sqrt{-\omega}(\{-\coth(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) \pm \operatorname{csch}(2\sqrt{-\omega}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}))^2, \quad (62)$$

$$g(x,t)=b_0-\frac{\sqrt{-\omega}}{2}(\{\tanh(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) + \coth(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}) \quad (63)$$

$$h(x,t) = 2b_0(b_0-\frac{\sqrt{-\omega}}{2}(\{\tanh(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) + \coth(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\})) -(b_0-\frac{\sqrt{-\omega}}{2}(\{\tanh(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha)) + \coth(\frac{\sqrt{-\omega}}{2}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha+2b_0t^\alpha))\}))^2 \quad (64)$$

Description of Sardar sub-equation method

In this portion, we mention the some basic steps of this method [31]. Assuming the non-linear PDE:

$$G(f, f_t, f_{xx}, f_{xt}, f f_{tt}, f_{xy}, \ldots)=0. \quad (65)$$

here $f=f(x,y,t)$ is a wave profile.
Using the wave transformation given as:

$$f(x,y,t)=F(\zeta), \zeta=\lambda x+\kappa y+\mu t \quad (66)$$

We obtain a non-linear ODE shown as:

$$Y(F, F'', FF'', F'F^2, \ldots)=0. \quad (67)$$

Consider Eq. (67) posses the solutions in the shown form:

$$F(\zeta)=\sum_{i=0}^{m} b_i\psi^i(\zeta). \quad (68)$$

where $\psi(\zeta)$ fulfill the ordinary differential equation given as:

$$\psi'(\zeta)=\sqrt{\sigma+\kappa\psi^2(\zeta)+\psi^4(\zeta)}. \quad (69)$$

here σ and κ are the parameters.
Inserting Eq. (68) into the Eq. (67) along Eq. (69) and summing up the co-efficients of every order of ψ^i. Substituting each of them equal to 0, we obtain a sets of algebraic equations containg b_i, λ, μ. Manipulating the gained a set of equations, we get the results for parameters.
Case 1: if $\kappa>0$ and $\sigma=0$, we get

$$\psi_1^{\pm}=\pm\sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \zeta), \quad (70)$$

$$\psi_2^{\pm}=\pm\sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \zeta), \quad (71)$$

where, $sech_{ab}(\zeta)=\frac{2}{ae^{\zeta}+be^{-\zeta}}$, $csch_{ab}(\zeta)=\frac{2}{ae^{\zeta}-be^{-\zeta}}$
Case 2: if $\kappa<0$ and $\sigma=0$, we get

$$\psi_3^{\pm}=\pm\sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \zeta), \quad (72)$$

$$\psi_4^{\pm}=\pm\sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \zeta), \quad (73)$$

where, $\sec_{ab}(\zeta)=\frac{2}{ae^{\iota\zeta}+be^{-\iota\zeta}}$, $\csc_{ab}(\zeta)=\frac{2\iota}{ae^{\iota\zeta}-be^{-\iota\zeta}}$

Case 3: if $\kappa < 0$ and $\sigma = \frac{\kappa^2}{4}$, we get

$$\psi_5^{\pm} = \pm\sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \zeta), \tag{74}$$

$$\psi_6^{\pm} = \pm\sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \zeta), \tag{75}$$

$$\psi_7^{\pm} = \pm\sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \zeta) \pm \iota\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \zeta)), \tag{76}$$

$$\psi_8^{\pm} = \pm\sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \zeta) \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \zeta)), \tag{77}$$

$$\psi_9^{\pm} = \pm\sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \zeta) + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \zeta)), \tag{78}$$

where, $\tanh_{ab}(\zeta) = \frac{ae^{\zeta}-be^{-\zeta}}{ae^{\zeta}+be^{-\zeta}}$, $\coth_{ab}(\zeta) = \frac{ae^{\zeta}+be^{-\zeta}}{ae^{\zeta}-be^{-\zeta}}$

Case 4: if $\kappa > 0$ and $\sigma = \frac{\kappa^2}{4}$, we get

$$\psi_{10}^{\pm} = \pm\sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \zeta), \tag{79}$$

$$\psi_{11}^{\pm} = \pm\sqrt{\frac{\kappa}{2}}\ \cot_{ab}(\sqrt{\frac{\kappa}{2}}\ \zeta), \tag{80}$$

$$\psi_{12}^{\pm} = \pm\sqrt{\frac{\kappa}{2}}(\tan_{ab}(\sqrt{2\kappa}\ \zeta) \pm \sqrt{ab}\ \sec_{ab}(\sqrt{2\kappa}\ \zeta)), \tag{81}$$

$$\psi_{13}^{\pm} = \pm\sqrt{\frac{\kappa}{2}}(\cot_{ab}(\sqrt{2\kappa}\ \zeta) \pm \sqrt{ab}\ \csc_{ab}(\sqrt{2\kappa}\ \zeta)), \tag{82}$$

$$\psi_{14}^{\pm} = \pm\sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \zeta) + \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \zeta)), \tag{83}$$

where, $\tan_{ab}(\zeta) = -\iota\frac{ae^{\iota\zeta}-be^{-\iota\zeta}}{ae^{\iota\zeta}+be^{-\iota\zeta}}$, $\cot_{ab}(\zeta) = \iota\frac{ae^{\iota\zeta}+be^{-\iota\zeta}}{ae^{\iota\zeta}-be^{-\iota\zeta}}$

New exact wave solutions through Sardar sub-equation technique

Eq. (68) reduces into shown shape for m = 1.

$$G(\zeta) = b_0 + b_1\psi(\zeta) \tag{84}$$

Putting Eq. (84) into Eq. (6) with Eq. (69). By solving the achieved system of equations by Mathematica tool, we obtain the shown results.
Set 1;

$$\left\{ b_0 = -\frac{i\sqrt{\kappa}}{\sqrt{2}}, b_1 = -1, \delta = i\sqrt{2\kappa} \right\} \tag{85}$$

Case 1

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \tag{86}$$

$$\begin{aligned} h(x,t) \ =\ & -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{87}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \tag{88}$$

$$\begin{aligned} h(x,t) \ =\ & -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{89}$$

Case 2

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \tag{90}$$

$$\begin{aligned} h(x,t) \ =\ & -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{91}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \tag{92}$$

$$\begin{aligned} h(x,t) \ =\ & -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{93}$$

Case 3

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})), \tag{94}$$

$$\begin{aligned} h(x,t) \ =\ & -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))^2, \end{aligned} \tag{95}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})), \tag{96}$$

$$\begin{aligned} h(x,t) \ =\ & -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))^2, \end{aligned} \tag{97}$$

$$\begin{aligned} g(x,t) =\ & -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \iota\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{98}$$

$$\begin{aligned} h(x,t) \ =\ & \kappa(-1 \mp (\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \iota\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \iota\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))))^2 \end{aligned} \tag{99}$$

$$\begin{aligned} g(x,t) =\ & -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{100}$$

$$\begin{aligned} h(x,t) \ =\ & \kappa(-1 \mp (\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))))^2 \end{aligned} \tag{101}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))$$

$$+\ \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \tag{102}$$

$$\begin{aligned} h(x,t) &= \kappa(-1\mp\frac{1}{2}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &+ \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &+ \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))))^2 \end{aligned} \tag{103}$$

Case 4

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})), \tag{104}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))^2, \end{aligned} \tag{105}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}\ \cot_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})), \tag{106}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}\ \cot_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}\ \cot_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))^2, \end{aligned} \tag{107}$$

$$\begin{aligned} g(x,t) &= -\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}(\tan_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &\pm\sqrt{ab}\ \sec_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))), \end{aligned} \tag{108}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{\kappa}(-i\sqrt{\kappa}\mp\sqrt{\kappa}(\tan_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &\pm\sqrt{ab}\ \sec_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}(\tan_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &\pm\sqrt{ab}\ \sec_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))), \end{aligned} \tag{109}$$

$$\begin{aligned} g(x,t) &= -\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}(\cot_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &\pm\sqrt{ab}\ \csc_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))), \end{aligned} \tag{110}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{\kappa}(-i\sqrt{\kappa}\mp\sqrt{\kappa}(\cot_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &\pm\sqrt{ab}\ \csc_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{2}}(\cot_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &\pm\sqrt{ab}\ \csc_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))))^2, \end{aligned} \tag{111}$$

$$\begin{aligned} g(x,t) &= -\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &+ \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))), \end{aligned} \tag{112}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &+ \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\mp\sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \\ &+ \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))), \end{aligned} \tag{113}$$

Set 2:

$$\left\{ b_0 = -\frac{i\sqrt{\kappa}}{\sqrt{2}}, b_1 = 1, \delta = i\sqrt{2}\sqrt{\kappa} \right\} \tag{114}$$

Case 1

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \tag{115}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{116}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \tag{117}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{118}$$

Case 2

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \tag{119}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{120}$$

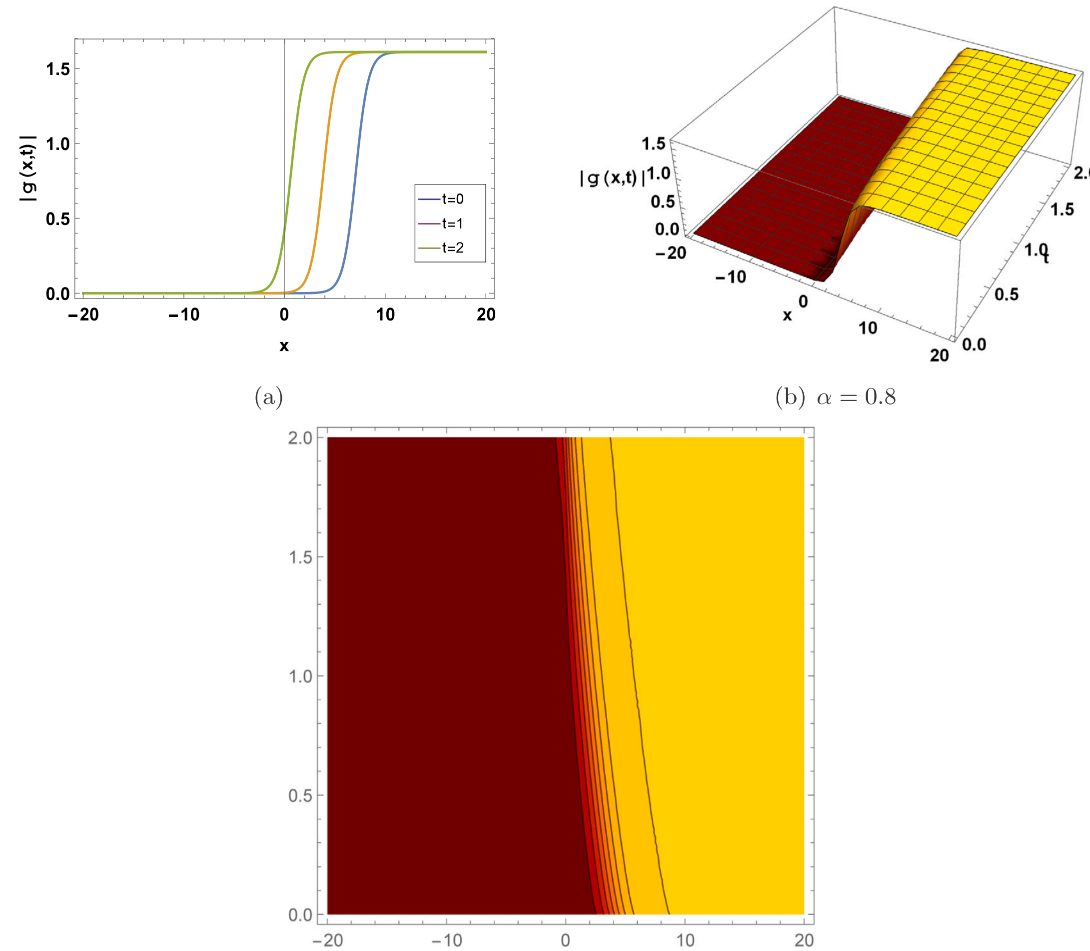
$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})) \tag{121}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))^2 \end{aligned} \tag{122}$$

Case 3

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})), \tag{123}$$

$$\begin{aligned} h(x,t) &= -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}}))) \\ &-(-\frac{i\sqrt{\kappa}}{\sqrt{2}}\pm\sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}-i\sqrt{2\kappa t^{\alpha}})))^2, \end{aligned} \tag{124}$$



(a) (b) $\alpha = 0.8$

(c)

Fig. 1. Plot for $|g(x,\ t)|$ shown in the Eq. (15) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})), \tag{125}$$

$$\begin{aligned} h(x,t) = & -i\sqrt{2\kappa}(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))^{2}, \end{aligned} \tag{126}$$

$$\begin{aligned} g(x,t) = & -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{127}$$

$$\begin{aligned} h(x,t) = & \ \kappa(-1 \pm (\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))))^{2} \end{aligned} \tag{128}$$

$$\begin{aligned} g(x,t) = & -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{129}$$

$$\begin{aligned} h(x,t) = & \ \kappa(-1 \pm (\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))))^{2} \end{aligned} \tag{130}$$

$$\begin{aligned} g(x,t) = & -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{131}$$

$$\begin{aligned} h(x,t) = & \ \kappa(-1 \pm \frac{1}{2}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})))) \\ & -(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) \\ & + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))))^{2} \end{aligned} \tag{132}$$

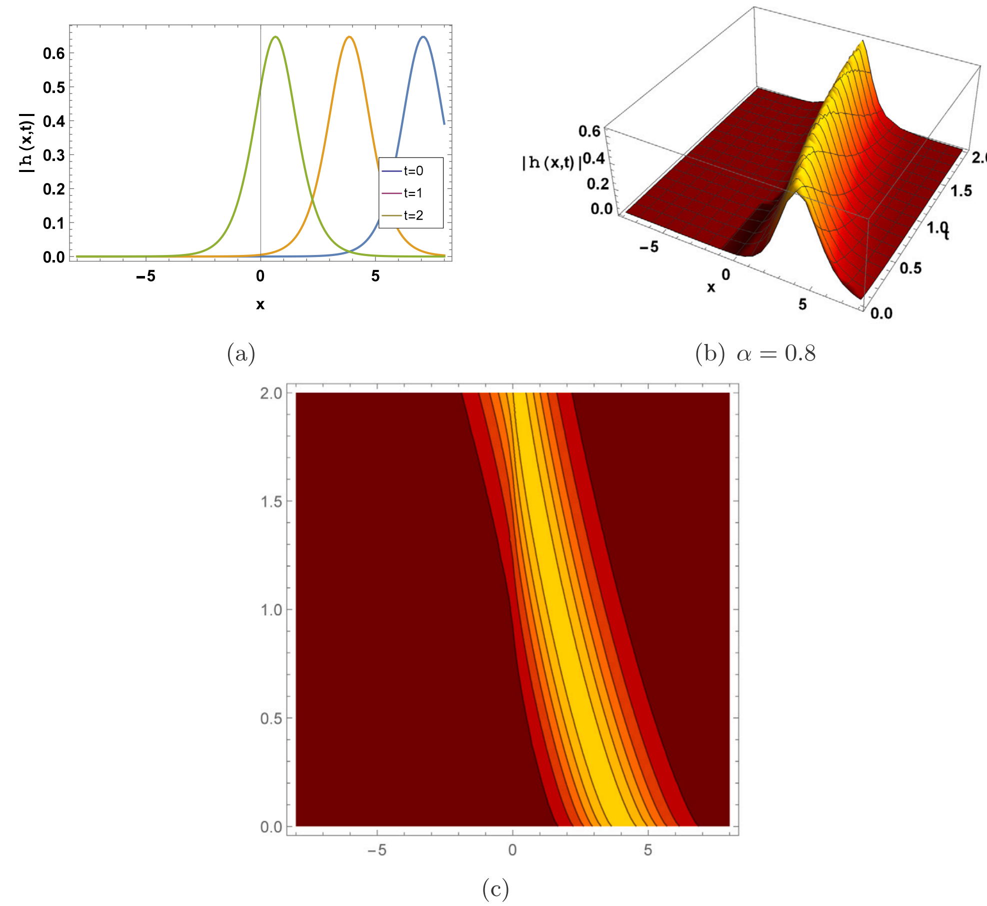


Fig. 2. Plot for $|h(x, t)|$ shown in the Eq. (16) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Case 4

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \tan_{ab}\left(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right), \tag{133}$$

$$h(x,t) = -i\sqrt{2\kappa}\left(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \tan_{ab}\left(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right) - \left(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \tan_{ab}\left(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right)^2, \tag{134}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \cot_{ab}\left(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right), \tag{135}$$

$$h(x,t) = -i\sqrt{2\kappa}\left(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \cot_{ab}\left(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right) - \left(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \cot_{ab}\left(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right)^2, \tag{136}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\left(\tan_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) \pm \sqrt{ab}\ \sec_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right), \tag{137}$$

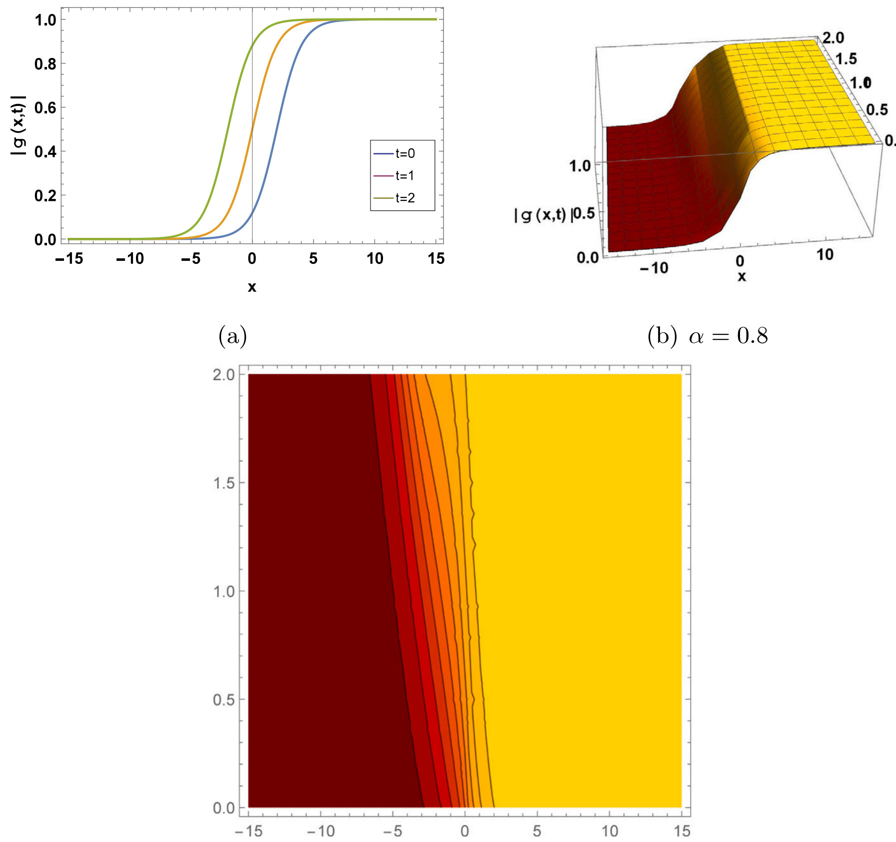
$$h(x,t) = -i\sqrt{\kappa}\left(-i\sqrt{\kappa} \pm \sqrt{\kappa}\left(\tan_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) \pm \sqrt{ab}\ \sec_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right)\right) - \left(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\left(\tan_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) \pm \sqrt{ab}\ \sec_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right)\right)^2, \tag{138}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\left(\cot_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) \pm \sqrt{ab}\ \csc_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right), \tag{139}$$

$$h(x,t) = -i\sqrt{\kappa}\left(-i\sqrt{\kappa} \pm \sqrt{\kappa}\left(\cot_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) \pm \sqrt{ab}\ \csc_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right)\right) - \left(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\left(\cot_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) \pm \sqrt{ab}\ \csc_{ab}\left(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right)\right)^2, \tag{140}$$

$$g(x,t) = -\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{8}}\left(\tan_{ab}\left(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) + \cot_{ab}\left(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right), \tag{141}$$

$$h(x,t) = -i\sqrt{2\kappa}\left(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{8}}\left(\tan_{ab}\left(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right) + \cot_{ab}\left(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})\right)\right)\right)$$



(a) (b) $\alpha = 0.8$ (c)

Fig. 3. Plot for $|g(x,\ t)|$ shown in Eq. (44) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$-(-\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}})) + \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} - i\sqrt{2\kappa t^{\alpha}}))))^2, \quad (142)$$

Set 3:

$$\left\{ b_0 = \frac{i\sqrt{\kappa}}{\sqrt{2}}, b_1 = -1, \delta = -i\sqrt{2}\sqrt{\kappa} \right\} \quad (143)$$

Case 1

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})) \quad (144)$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})))^2 \quad (145)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})) \quad (146)$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})))^2 \quad (147)$$

Case 2

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})) \quad (148)$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})))^2 \quad (149)$$

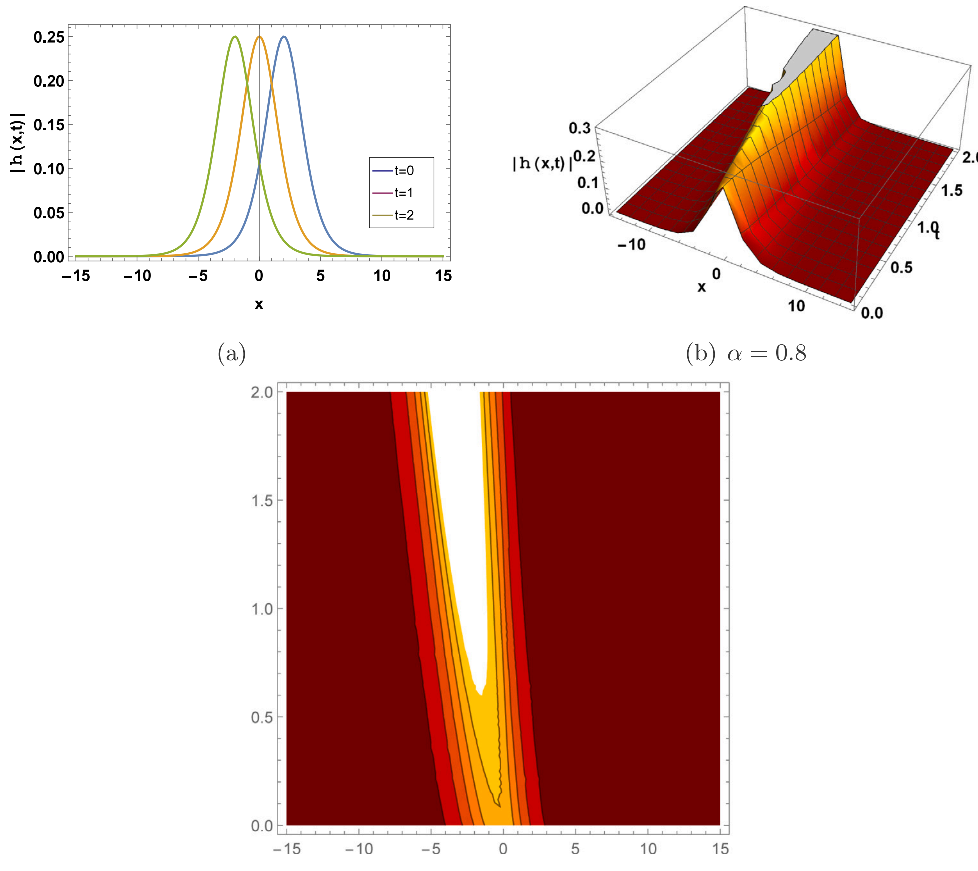
$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})) \quad (150)$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})))^2 \quad (151)$$

Case 3

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})), \quad (152)$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha} + i\sqrt{2\kappa t^{\alpha}})))^2, \quad (153)$$



(a) (b) $\alpha = 0.8$

(c)

Fig. 4. Plot for $|h(x, t)|$ shown in Eq. (45) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})), \tag{154}$$

$$\begin{aligned} h(x,t) &= i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) \\ &\quad -(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))^2, \end{aligned} \tag{155}$$

$$\begin{aligned} g(x,t) &= \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{156}$$

$$\begin{aligned} h(x,t) &= -\kappa(1 \mp (\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))) \\ &\quad -(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))))^2 \end{aligned} \tag{157}$$

$$\begin{aligned} g(x,t) &= \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{158}$$

$$\begin{aligned} h(x,t) &= -\kappa(1 \mp (\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))) \\ &\quad -(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))))^2 \end{aligned} \tag{159}$$

$$\begin{aligned} g(x,t) &= \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) \end{aligned} \tag{160}$$

$$\begin{aligned} h(x,t) &= -\kappa(1 \mp \frac{1}{2}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))) \\ &\quad -(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \\ &\quad + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))))^2 \end{aligned} \tag{161}$$

Case 4

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})), \tag{162}$$

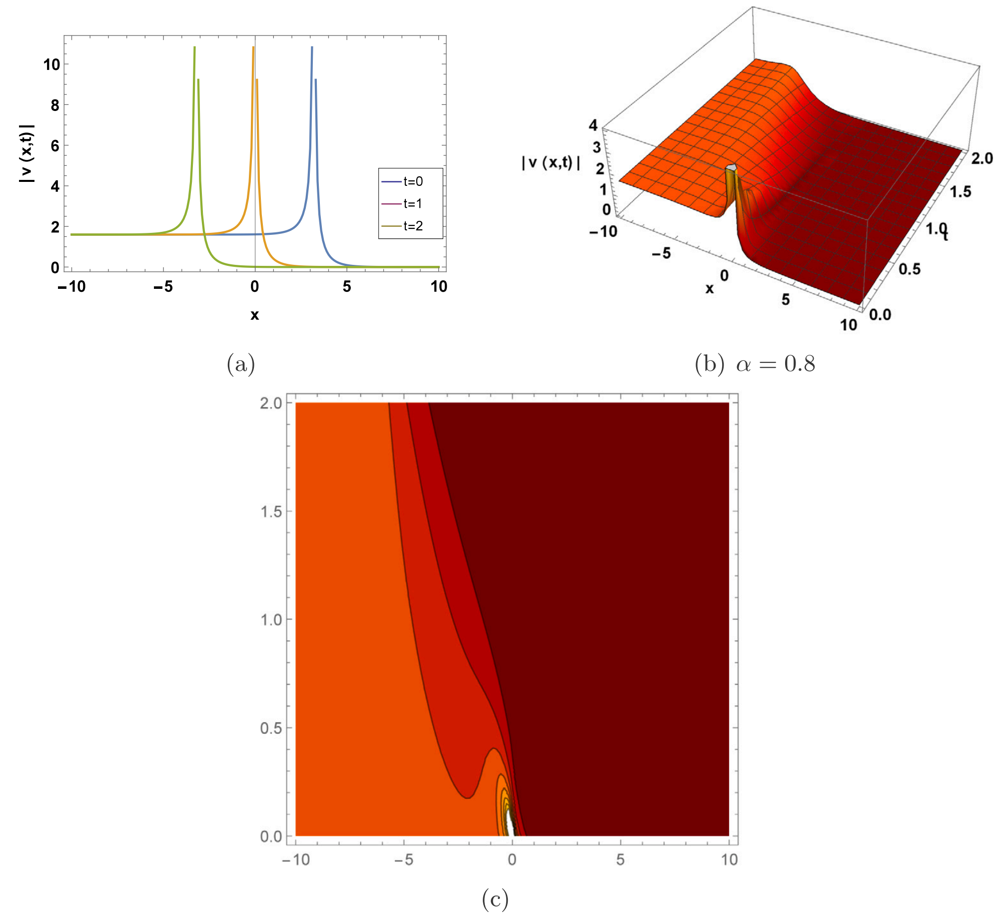


Fig. 5. Plot for $|g(x,\ t)|$ shown in Eq. (63) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$h(x,t) = i\sqrt{2\kappa}\Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\ \tan_{ab}\Big(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big) - \Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\ \tan_{ab}\Big(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)^2, \quad (163)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\ \cot_{ab}\Big(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big), \quad (164)$$

$$h(x,t) = i\sqrt{2\kappa}\Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\ \cot_{ab}\Big(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big) - \Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\ \cot_{ab}\Big(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)^2, \quad (165)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\Big(\tan_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) \pm \sqrt{ab}\ \sec_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big), \quad (166)$$

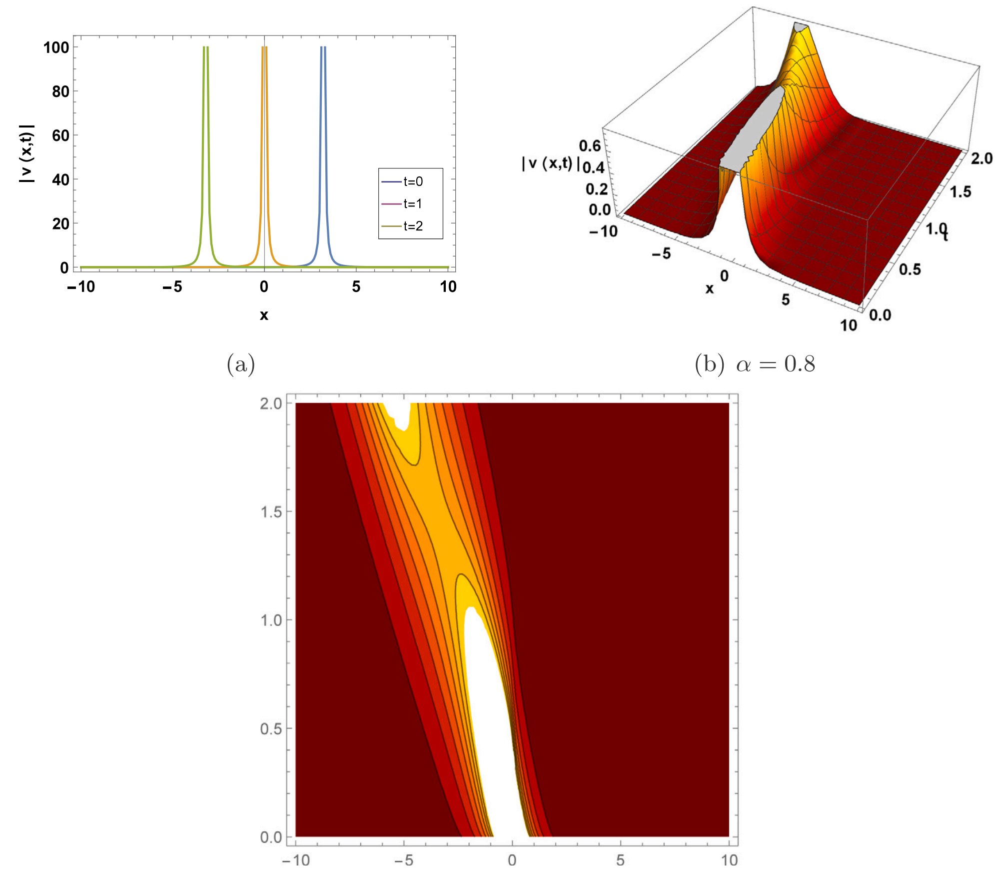
$$h(x,t) = i\sqrt{\kappa}\Big(i\sqrt{\kappa} \mp \sqrt{\kappa}\Big(\tan_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) \pm \sqrt{ab}\ \sec_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)\Big) - \Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\Big(\tan_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) \pm \sqrt{ab}\ \sec_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)\Big)^2, \quad (167)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\Big(\cot_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) \pm \sqrt{ab}\ \csc_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big), \quad (168)$$

$$h(x,t) = i\sqrt{\kappa}\Big(i\sqrt{\kappa} \mp \sqrt{\kappa}\Big(\cot_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) \pm \sqrt{ab}\ \csc_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)\Big) - \Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{2}}\Big(\cot_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) \pm \sqrt{ab}\ \csc_{ab}\Big(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)\Big)^2, \quad (169)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{8}}\Big(\tan_{ab}\Big(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) + \cot_{ab}\Big(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big), \quad (170)$$

$$h(x,t) = i\sqrt{2\kappa}\Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{8}}\Big(\tan_{ab}\Big(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) + \cot_{ab}\Big(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)\Big) - \Big(\frac{i\sqrt{\kappa}}{\sqrt{2}} \mp \sqrt{\frac{\kappa}{8}}\Big(\tan_{ab}\Big(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big) + \cot_{ab}\Big(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})\Big)\Big)\Big)^2, \quad (171)$$



(a) (b) $\alpha = 0.8$ (c)

Fig. 6. Plot $|h(x, t)|$ shown in the Eq. (64) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Set 4:

$$\left\{ b_0 = \frac{i\sqrt{\kappa}}{\sqrt{2}}, b_1 = 1, \delta = -i\sqrt{2}\sqrt{\kappa} \right\} \tag{172}$$

Case 1

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})) \tag{173}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ sech_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})))^2 \tag{174}$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})) \tag{175}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\kappa ab}\ csch_{ab}(\sqrt{\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})))^2 \tag{176}$$

Case 2

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})) \tag{177}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ \sec_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})))^2 \tag{178}$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})) \tag{179}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\kappa ab}\ \csc_{ab}(\sqrt{-\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})))^2 \tag{180}$$

Case 3

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})), \tag{181}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \tanh_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})))^2, \tag{182}$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})), \tag{183}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^\alpha + i\sqrt{2\kappa t^\alpha})))$$

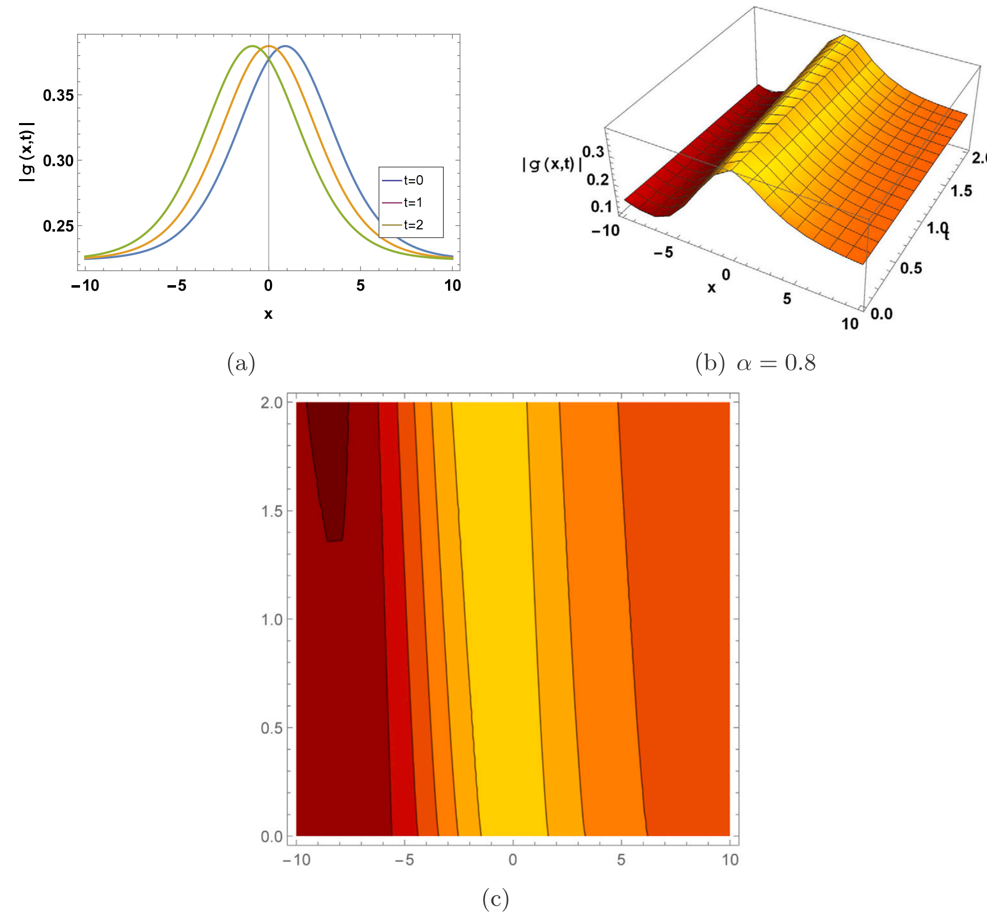


Fig. 7. Plot for $|g(x,\ )|$ shown in Eq. (86) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$-(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}\ \coth_{ab}(\sqrt{-\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))^2, \quad (184)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) \quad (185)$$

$$h(x,t) = -\kappa(1 \pm (\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\tanh_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \pm \imath\sqrt{ab}\ sech_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))))^2 \quad (186)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) \quad (187)$$

$$h(x,t) = -\kappa(1 \pm (\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{2}}(\coth_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) \pm \sqrt{ab}\ csch_{ab}(\sqrt{-2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))))^2 \quad (188)$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) \quad (189)$$

$$h(x,t) = -\kappa(1 \pm \frac{1}{2}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{-\frac{\kappa}{8}}(\tanh_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})) + \coth_{ab}(\sqrt{-\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))))^2 \quad (190)$$

Case 4

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})), \quad (191)$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \tan_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa t^{\alpha}})))^2, \quad (192)$$

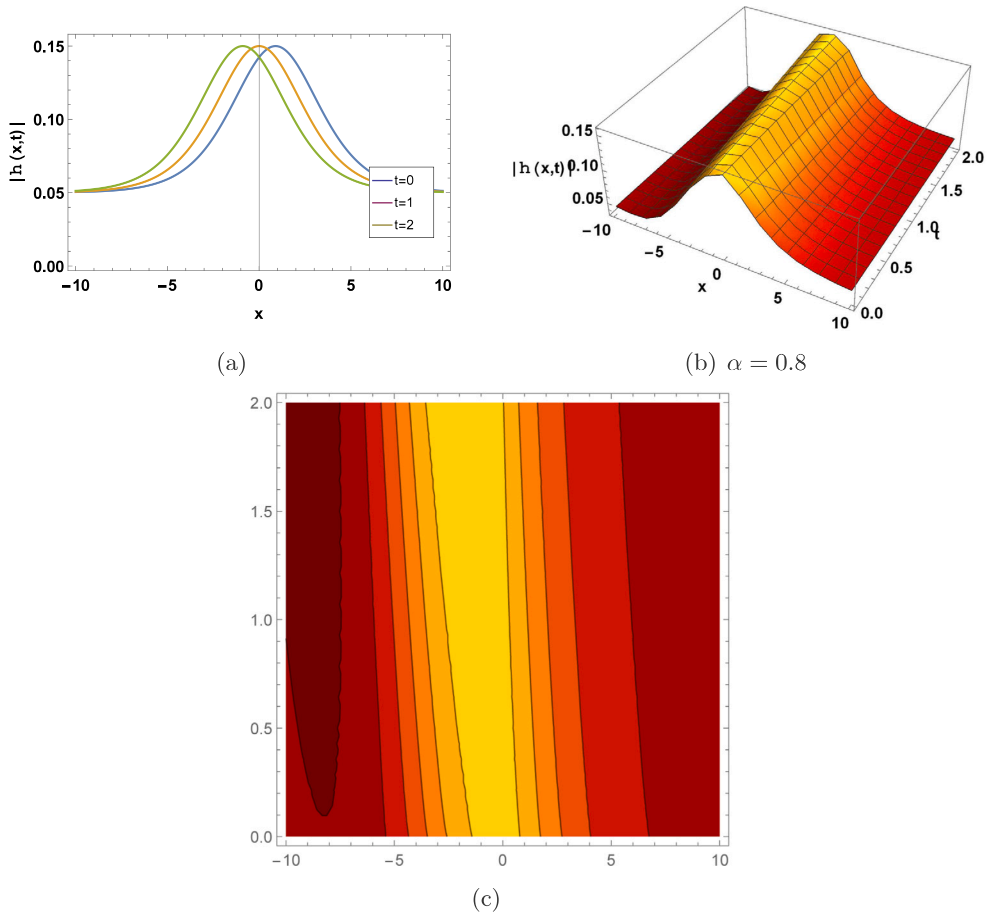


Fig. 8. Plot for $|h(x,t)|$ shown in the Eq. (87) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \cot_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})), \tag{193}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \cot_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha}))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}\ \cot_{ab}(\sqrt{\frac{\kappa}{2}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})))^{2}, \tag{194}$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}(\tan_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) \pm \sqrt{ab}\ \sec_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha}))), \tag{195}$$

$$h(x,t) = i\sqrt{2\kappa}(i\sqrt{\kappa} \pm \sqrt{\kappa}(\tan_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) \pm \sqrt{ab}\ \sec_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}(\tan_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) \pm \sqrt{ab}\ \sec_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha}))))^{2}, \tag{196}$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}(\cot_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) \pm \sqrt{ab}\ \csc_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha}))), \tag{197}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}(\cot_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) \pm \sqrt{ab}\ \csc_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{2}}(\cot_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) \pm \sqrt{ab}\ \csc_{ab}(\sqrt{2\kappa}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha}))))^{2}, \tag{198}$$

$$g(x,t) = \frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) + \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha}))), \tag{199}$$

$$h(x,t) = i\sqrt{2\kappa}(\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) + \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})))) - (\frac{i\sqrt{\kappa}}{\sqrt{2}} \pm \sqrt{\frac{\kappa}{8}}(\tan_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha})) + \cot_{ab}(\sqrt{\frac{\kappa}{8}}\ \frac{\Gamma(1+\Upsilon)}{\alpha}(x^{\alpha}+i\sqrt{2\kappa}t^{\alpha}))))^{2}, \tag{200}$$

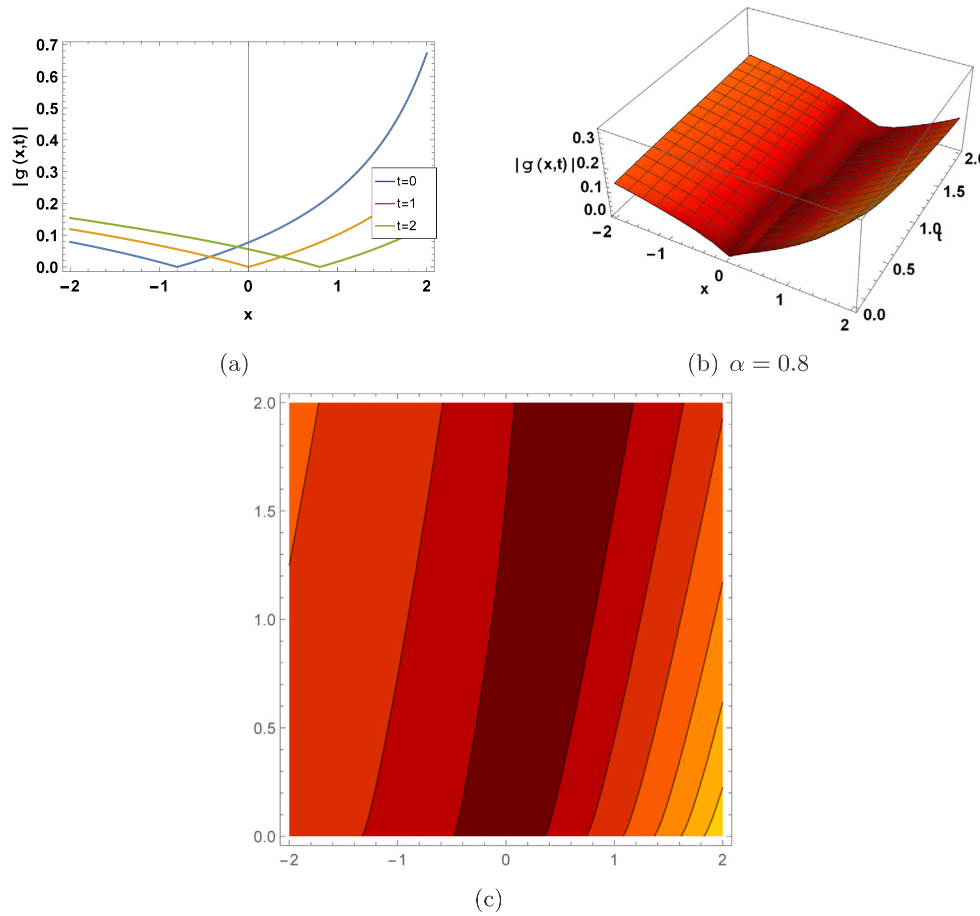


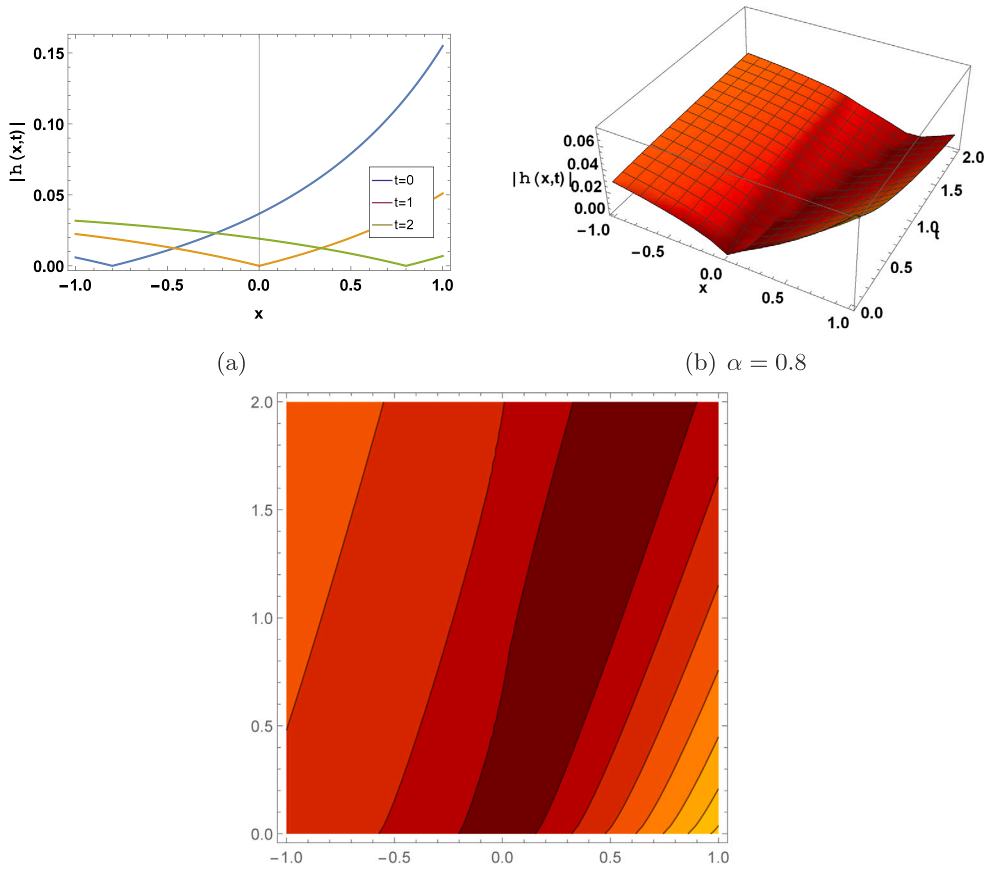
Fig. 9. Plot for $|g(x,\ t)|$ shown in the Eq. (166) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Graphically interpretation of results

Here, we graphically explain the some of our achieved new soliton solutions through figures. In Fig. 1, we represent the plot of $|g(x,\ t)|$ shown in the Eq. (15) for $d = 5$; $\beta_0 = 0.5$; $\beta_1 = 0.001$; $\Upsilon = 1$; and $x \in (-20, 20)$ in 2-D at $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $t \in (0, 2)$. In Fig. 2, we represent the plot of $|h(x,\ t)|$ shown in the Eq. (16) for $d = 5$; $\beta_0 = 0.5$; $\beta_1 = 0.001$; $\Upsilon = 1$; and $x \in (-8, 8)$ in 2-D at $\alpha = 1$, blue curve drawn at $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $0 < t < 2$. In Fig. 3, we represent the plot of $|g(x,\ t)|$ shown in the Eq. (44) for $\Upsilon = 1$; $b_0 = 0.5$ and $x \in (-15, 15)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $t \in (0, 2)$. In Fig. 4, we represent the plot of $|h(x,\ t)|$ shown in the Eq. (45) for $\Upsilon = 1$; $b_0 = 0.5$ and $x \in (-15, 15)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha 0.8$ and $t \in (0, 2)$. In Fig. 5, we represent the plot of $|g(x,\ t)|$ shown in the Eq. (63) for $\Upsilon = 1$; $b_0 = 0.8$; and $(-10, 10)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $t \in (0, 2)$. In Fig. 6, we represent a plot of $|h(x,\ t)|$ shown in the Eq. (64) for $\Upsilon = 1$; $b_0 = 0.8$ and $x \in (-10, 10)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $t \in (0, 2)$. In Fig. 7, we represent the plot of $|g(x,\ t)|$ shown in the Eq. (86) for $\Upsilon = 1$; $\kappa = 0.1$ and $x \in (-10, 10)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $t \in (0, 2)$. In Fig. 8, we represent the plot of $|h(x,\ t)|$ shown in the Eq. (87) for $\Upsilon = 1$; $\kappa = 0.1$ and $x \in (-10, 10)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn at $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $t \in (0, 2)$. In Fig. 9, we represent the plot of $|g(x,\ t)|$ shown in the Eq. (166) for $\Upsilon = 1$; $\kappa = 0.08$ and $x \in (-2, 2)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha = 0.8$ and $t \in (0, 2)$. In Fig. 10, we represent the plot of $|h(x,\ t)|$ shown in the Eq. (167) for $\Upsilon = 1$; $\kappa = 0.08$ and $x \in (-1, 1)$ in 2-D for $\alpha = 1$, blue curve drawn for $t = 0$, orange curve drawn for $t = 1$, green curve drawn for $t = 2$, in 3-D for $\alpha = 0.8$ and $t \in (0, 2)$, and in contour for $\alpha 0.8$ and $t \in (0, 2)$.

Conclusion

Overall, we are succeed to gain the new soliton solutions for (1+1)-dimensional non-linear Kaup–Boussinesq system and understand the fruitful techniques for handling non-linear fractional PDEs. This paper explains the successful uses of $\exp_a$ function and modified simplest equation and Sardar sub-equation techniques. The achieved solutions having dark, bright, dark-bright, periodic and other solitons. The solutions are verified as well as explained through 2-D, 3-D and contour plots using Mathematica tool. The gained results are fruitful for further research about our governing model. The attained solutions are very



(a) (b) $\alpha = 0.8$

(c)

Fig. 10. Plot for $|h(x,\ t)|$ shown in the Eq. (167) in 2-D, 3-D and contour. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

helpful to study the long waves in shallow water. Results may be used in many area of telecommunications, optical fibers, engineering and others. Due to the use of truncated M-fractional derivative, our results are very close to the approximate results. The $\exp_a$ function, modified simplest equation and Sardar sub-equation techniques are represented straight forward and useful techniques to deal with nonlinear fractional PDEs. The proposed techniques is highly efficient and can also be utilized to solve higher-order nonlinear partial differential equations in fields such as fluid mechanics, plasma physics, and fiber optics.

CRediT authorship contribution statement

Asim Zafar: Project administration, Validation, Writing – review & editing. **M. Raheel:** Investigation, Methodology, Writing – original draft. **Ali M. Mahnashi:** Methodology, Software, Writing – original draft. **Ahmet Bekir:** Formal analysis, Supervision, Writing – review & editing. **Mohamed R. Ali:** Conceptualization, Funding acquisition, Writing – original draft. **A.S. Hendy:** Data curation, Investigation, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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